Z10) $\lim\limits_{x\to 0}\frac{1}{x}\left(\frac{1}{x}-\operatorname{cotg} x\right)=\lim\limits_{x\to 0}\frac{\sin x - x\cos x}{x^2\sin x}\overset{(*)}{=}\lim\limits_{x\to 0}\frac{\frac{x^3}{3}+o(x^3)}{x^3}\cdot\frac{x}{\sin x}=\underline{\underline{\frac{1}{3}}}$

$(*)\ \sin x = x-\frac{x^3}{6}+o(x^3)$

$\cos x = 1-\frac{x^2}{2}+o(x^2) \Rightarrow x\cos x = x-\frac{x^3}{2}+o(x^3)\Big\} \Rightarrow \sin x - x\cos x = \frac{x^3}{3}+o(x^3)$

Z16) $\operatorname{tg}(y)=\frac{\sin y}{\cos y}=\frac{x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{7!}+o(x^8)}{1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{6!}+o(x^7)}$

$\parallel$

$Ax+Bx^3+Cx^5+Dx^7+o(x^8)$

$\Rightarrow x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{7!}+o(x^8)=\left(Ax+Bx^3+Cx^5+Dx^7+o(x^8)\right)\left(1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{6!}+o(x^7)\right)$

~~jednoznačnost Taylo. pol.~~

$=\underbrace{Ax+x^3\left(-\tfrac{1}{2}A+B\right)+x^5\left(\frac{A}{24}-\frac{B}{2}+C\right)+x^7\left(-\frac{A}{6!}+\frac{B}{24}-\frac{C}{2}+D\right)}+o(x^8)$

Podle věty o jednoznačnosti Taylo. polynomu je polynom zde roven Taylo. polynomu fce sin v 0 řádu 8. Polynomy se rovnají $\Leftrightarrow$ se rovnají koef. $\Rightarrow \underline{A=1}$

$-\frac{A}{2}+B=-\frac{1}{6} \Downarrow B=\frac{1}{3}$

$\frac{A}{24}-\frac{B}{2}+C=\frac{1}{120}\Rightarrow C=\frac{2}{15}$

$-\frac{A}{6!}+\frac{B}{24}-\frac{C}{2}+D=-\frac{1}{7!}\Rightarrow$

$-1=-7+70-\underbrace{\frac{5040}{15}}_{336}+D.7!\Rightarrow D=\frac{272}{7!}=\frac{17}{315}$

$\Rightarrow \operatorname{tg}(x)=x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7+o(x^8)$

$\operatorname{tg}(\sin x)=\sin x+\frac{(\sin x)^3}{3}+\frac{2}{15}\sin^5 x+\frac{17}{315}\sin^7 x+o\left((\sin x)^8\right)\overset{(+)}{=}$

$x-\frac{x^3}{6}+\frac{x^5}{5!}-\frac{x^7}{7!}+o(x^8)+\frac{x^3}{3}-\frac{x^5}{6}+\frac{x^7 13}{360}+o(x^8)+\frac{2}{15}x^5-\frac{x^7}{9}+o(x^8)+\frac{17}{315}x^7+o(x^8)=(**)$

$(+)\ \lim\limits_{x\to 0}\frac{\sin x}{x}=1\in\mathbb{R}\setminus\{0\}\Rightarrow o\left((\sin x)^n\right)=o(x^n), n\in\mathbb{N}$

$\sin^3 x=\left(x-\frac{x^3}{6}+\frac{x^5}{5!}+o(x^6)\right)^3=x^3-\frac{x^5}{2}+x^7\underbrace{\left(\frac{3}{5!}+\frac{3}{6^2}\right)}_{\frac{13}{120}}+o(x^8)$

$\sin^5 x=\left(x-\frac{x^3}{6}+o(x^4)\right)^5=x^5-\frac{5}{6}x^7+o(x^8)$

$\sin^7 x=\left(x+o(x^2)\right)^7=x^7+o(x^8)$

$(**)=x+\frac{x^3}{6}+x^5\underbrace{\left(\frac{1}{5!}-\frac{1}{6}+\frac{2}{15}\right)}_{-\frac{1}{40}}+x^7\underbrace{\left(-\frac{1}{7!}+\frac{13}{360}-\frac{1}{9}+\frac{17}{315}\right)}_{\frac{-1+13.14-560+272}{7!}=-\frac{107}{7!}}+o(x^8)$

$\operatorname{tg}^3 x=\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+o(x^6)\right)^3=x^3+x^5+x^7\underbrace{\left(\frac{2}{5}+\frac{1}{3}\right)}_{\frac{11}{15}}+o(x^8)$

$\operatorname{tg}^5 x=\left(x+\frac{x^3}{3}+o(x^4)\right)^5=x^5+\frac{5}{3}x^7+o(x^8)$ $\quad\lim\limits_{x\to 0}\frac{\operatorname{tg} x}{x}=1\in\mathbb{R}\setminus\{0\}$

$\operatorname{tg}^7 x=\left(x+o(x^2)\right)^7=x^7+o(x^8)$

$\sin(\operatorname{tg} x)=\operatorname{tg} x-\frac{\operatorname{tg}^3 x}{6}+\frac{\operatorname{tg}^5 x}{120}-\frac{\operatorname{tg}^7 x}{7!}+\overbrace{o(\operatorname{tg}^8 x)}^{o(x^8)\Leftarrow}=x+\frac{x^3}{3}+\frac{2}{15}x^5+\frac{17}{315}x^7$

$-\frac{x^3}{6}-\frac{x^5}{6}-\frac{x^7.11}{90}+\frac{x^5}{120}+\frac{x^7}{72}-\frac{x^7}{7!}+o(x^8)=x+\frac{x^3}{6}+x^5\underbrace{\left(\frac{2}{15}-\frac{1}{6}+\frac{1}{120}\right)}_{-\frac{1}{40}}+x^7\cdot$

$$\underbrace{\left(-\frac{1}{7!}+\frac{1}{72}-\frac{11}{90}+\frac{17}{315}\right)}$$

$$\frac{-1+70-616+272}{7!}=\frac{-275}{7!}=-\frac{55}{1008}$$

$$\lim_{x\to 0}\frac{\operatorname{tg}(\sin x)-\sin(\operatorname{tg}x)}{x^7}=\frac{-107+275}{7!}=\frac{168}{7!}=\underline{\underline{\frac{1}{30}}}$$

$\lim_{x\to 0}\frac{\sin x}{x}=1\in\mathbb{R}\setminus\{0\}\Rightarrow o(x^6)$

Př: Taylor. arcsin x řádu 6 v 0: $\arcsin x=\underline{Ax+Bx^3+Cx^5}+o(x^6)$

$$\boxed{\begin{aligned}\sin x&=x-\frac{x^3}{6}+\frac{x^5}{120}+o(x^6)\\ \sin^3 x&=x^3-\frac{x^5}{2}+o(x^6)\\ \sin^5 x&=x^5+o(x^6)\end{aligned}}$$

$$x=\arcsin(\sin x)=A\sin x+B\sin^3 x+C\sin^5 x+o(\sin^6 x)$$
$$=A\left(x-\frac{x^3}{6}+\frac{x^5}{120}\right)+B\left(x^3-\frac{x^5}{2}\right)+Cx^5+o(x^6)=Ax+$$
$$x^3\left(-\frac{A}{6}+B\right)+x^5\left(\frac{A}{120}-\frac{B}{2}+C\right)+o(x^6)$$

Z věty o jednoznačnosti Taylorova rozvoje: $\underline{A=1}$; $-\frac{A}{6}+B=0$; $\frac{A}{120}-\frac{B}{2}+C=0$

$$\Rightarrow \underline{B=\frac{1}{6}};\ \underline{C=\frac{3}{40}}.$$

Př: $$\lim_{x\to 0}\frac{\sin^2(\sin x)-2\sin(\sin^2 x)+\sin(\sin x^2)}{x^6}=\lim_{x\to 0}\frac{x^2(1-2+1)+x^4\left(-\frac{2}{3}+\frac{2}{3}\right)+x^6\left(\overset{\frac{2}{9}}{\frac{28+22-30}{90}}\right)+o(x^6)}{x^6}=\underline{\underline{\frac{2}{9}}}$$

$\lim_{x\to 0}\frac{\sin x}{x}=1\in\mathbb{R}\setminus\{0\}\Rightarrow o(x^6)$

$$\sin(\sin x)=\sin x-\frac{\sin^3 x}{6}+\frac{\sin^5 x}{5!}+o(\sin^6 x)=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^3}{6}+\frac{x^5}{12}+\frac{x^5}{120}+o(x^6)$$
$$=x-\frac{x^3}{3}+\frac{x^5}{10}+o(x^6)$$

$$\Rightarrow \sin^2(\sin x)=x^2-\frac{2x^4}{3}+x^6\left(\frac{1}{9}+\frac{1}{5}\right)+o(x^6)$$

$$\sin(\sin x^2)=x^2-\frac{x^6}{3}+o(x^6)\qquad \sin^6 x=x^6+o(x^6)$$

$$\sin^2 x=x^2-\frac{x^4}{3}+x^6\left(\frac{1}{36}+\frac{1}{60}\right)+o(x^6)$$

$o(x^6)$

$$\Rightarrow \sin(\sin^2 x)=\sin^2 x-\frac{\sin^6 x}{6}+o(\sin^8 x)=x^2-\frac{x^4}{3}+x^6\underbrace{\left(\frac{1}{36}+\frac{1}{60}-\frac{1}{6}\right)}_{-\frac{11}{90}}+o(x^6)$$